Exercise 26

Find the line through $(3,1,-2)$ that intersects and is perpendicular to the line $x=-1+t$, $y=-2+t$, $z=-1+t$ [Hint: If (x_0,y_0,z_0) is the point of intersection, find its coordinates.]

Solution

The equation for the line we're trying to find is

$$\mathbf{r}(t)=\mathbf{m}t+\mathbf{b}.$$

We require that the direction vector $\mathbf{m}$ be orthogonal to the direction vector of the given line.

$$(-1+t,-2+t,-1+t)=(t,t,t)+(-1,-2,-1)=(1,1,1)t+(-1,-2,-1)$$

In other words, the dot product of $\mathbf{m}$ and $(1,1,1)$ is zero.

$$\mathbf{m}\cdot(1,1,1)=0$$

$$\begin{aligned}(m_x,m_y,m_z)\cdot(1,1,1)&=0\\ m_x+m_y+m_z&=0 \qquad (1)\end{aligned}$$

The sum of the components must be zero. Make it so that the line goes through $(3,1,-2)$ at $t=0$.

$$\mathbf{r}(0)=(3,1,-2)=\mathbf{b}$$

As a result,

$$\begin{aligned}\mathbf{r}(t)&=(m_x,m_y,m_z)t+(3,1,-2)\\ &=(m_xt,m_yt,m_zt)+(3,1,-2)\\ &=(m_xt+3,m_yt+1,m_zt-2).\end{aligned}$$

Also, make it so that this line and the given line intersect at $t=t_0$.

$$\mathbf{r}(t_0)=(m_xt_0+3,m_yt_0+1,m_zt_0-2)=(-1+t_0,-2+t_0,-1+t_0)$$

Match the components to obtain a system of equations.

$$\begin{aligned}m_xt_0+3&=-1+t_0\\ m_yt_0+1&=-2+t_0\\ m_zt_0-2&=-1+t_0\end{aligned}$$

Together with equation (1) there are four equations and four unknowns. Solving it yields

$$m_x=-1 \quad\text{and}\quad m_y=-\frac{1}{2} \quad\text{and}\quad m_z=\frac{3}{2} \quad\text{and}\quad t_0=2.$$

Therefore, an equation for the desired line is

$$\mathbf{r}(t)=\left(-1,-\frac{1}{2},\frac{3}{2}\right)t+(3,1,-2).$$